Random Effects Models for Panel Data

November 24, 2014

Notation

Recall our estimating equation

$$y_{it} = \mathbf{x}_{it}\beta + c_i + \epsilon_{it} \tag{1}$$

where

$$\mathbf{x}_{it} = \begin{bmatrix} 1 & x_{it1} & x_{it2} & \dots & x_{itK} \end{bmatrix}_{1\times(K+1)} \tag{2}$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}_{(K+1)\times 1} \tag{3}$$

and we can define $v_{it} = c_i + \epsilon_{it}$. The components of this overall model error are

$$c_i \sim N(0, \sigma_c^2 I) \tag{4}$$

$$\epsilon_{it} \sim N(0, \sigma_\epsilon^2 I) \tag{5}$$

Some Real Data

person	year	income	age	sex
1	2003	1500	27	1
1	2004	1700	28	1
1	2005	2000	29	1
2	2003	2100	41	2
2	2004	2100	42	2
2	2005	2200	43	2

Assumption 1: Restrictions on the error structure

$$\Sigma = E[v_i' v_i] = \sigma_\epsilon^2 I + \sigma_c^2 \psi\psi' \tag{6}$$

$$= \begin{bmatrix} \sigma_\epsilon^2 + \sigma_c^2 & \sigma_c^2 & \dots & \sigma_c^2 \\ \sigma_c^2 & \sigma_\epsilon^2 + \sigma_c^2 & \dots & \sigma_c^2 \\ \vdots & & \ddots & \vdots \\ \sigma_c^2 & \dots & & \sigma_\epsilon^2 + \sigma_c^2 \end{bmatrix}_{T \times T} \tag{7}$$

where ψ is a $T \times 1$ vector of 1's.

A closer looks at Σ

On diagonal elements are

$$E[(v_{it}-0)^2] = E(v_{it}^2) \tag{8}$$
$$= E[(\epsilon_{it}+c_i)^2] \tag{9}$$
$$= E[\epsilon_{it}^2 + 2\epsilon_{it}c_i + c_i^2] \tag{10}$$
$$= E[\epsilon_{it}^2 + c_i^2] \tag{11}$$
$$= \sigma_\epsilon^2 + \sigma_c^2 \tag{12}$$

Off diagonal elements are

$$
\begin{aligned}
E[(v_{it}-0)(v_{it+1}-0)] &= E[(v_{it}v_{it+1})] && (13)\\
&= E[(\epsilon_{it}+c_i)(\epsilon_{it+1}+c_i)] && (14)\\
&= E[\epsilon_{it}\epsilon_{it+1}+\epsilon_{it}c_i+\epsilon_{it+1}c_i+c_i^2] && (15)\\
&= \sigma_c^2 && (16)
\end{aligned}
$$

Important Point: No elements in sigma are indexed by i: Σ is assumed to describe the covariances of the errors *within* each and every cross-section observation.

The Full Variance Covariance Matrix for the Errors in the Model

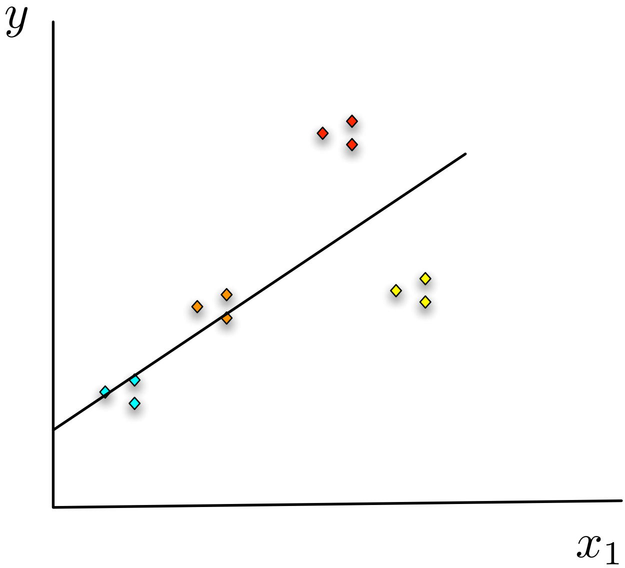
$$\Omega = \begin{bmatrix} \Sigma_{T\times T} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Sigma_{T\times T} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \Sigma_{T\times T} \end{bmatrix}_{NT\times NT} \tag{17}$$

Notice this structure allows errors to be correlated within each cross section unit, but uncorrelated across units.

Example

Rob's error in 2001 can be correlated with Rob's error in 2002 (via c_{rob}), but will not be correlated with Kim's error in 2001 or 2002, since c_{rob} is independent of c_{kim}.

Correlations Across Errors *within* Cross Section Unit



Assumption 2: The Rank Condition

$$rank\left(E(\mathbf{x}'\mathbf{\Omega}^{-1}\mathbf{x})\right) = K + 1 \tag{18}$$

The importance of this condition stems from our need to invert $\mathbf{x}'\mathbf{\Omega}^{-1}\mathbf{x}$.

Parameter Estimates

The Random Parameters Approach uses Ω to weight each observation into similar groups. These similar groups are the cross section units. Since Ω is block diagonal, the contribution of each cross section unit's sum of squares on the overall estimate of β is weighted by Σ. The Pooled OLS estimate assumes this weight is I, which is bound to be wrong because of the presence of unobserved heterogeneity.

Estimator:

$$\mathbf{b_{RE}} = (\mathbf{x}'\hat{\mathbf{\Omega}}^{-\mathbf{1}}\mathbf{x})^{-1}\mathbf{x}'\hat{\mathbf{\Omega}}^{-\mathbf{1}}\mathbf{y} \tag{19}$$

To implement this, we first need to estimate $\hat{\Omega}$

Estimating $\boldsymbol{\Sigma}$

$\boldsymbol{\Sigma}$ is comprised of two pieces of information:

$$\Sigma = E[v_i' v_i] = \sigma_\epsilon^2 I + \sigma_c^2 \psi \psi' \tag{20}$$

$$= \begin{bmatrix} \sigma_\epsilon^2 + \sigma_c^2 & \sigma_c^2 & \dots & \sigma_c^2 \\ \sigma_c^2 & \sigma_\epsilon^2 + \sigma_c^2 & \dots & \sigma_c^2 \\ \vdots & & \ddots & \vdots \\ \sigma_c^2 & \dots & & \sigma_\epsilon^2 + \sigma_c^2 \end{bmatrix}_{T \times T} \tag{21}$$

Strategy:

Step 1. Using pooled OLS calculate residuals:

$$\hat{\mathbf{v}} = \mathbf{y} - \mathbf{x}\mathbf{b}_{\mathbf{ols}} = \mathbf{y} - \mathbf{x}(\mathbf{x}'\mathbf{x})^{-\mathbf{1}}\mathbf{x}'\mathbf{y} \tag{22}$$

Intuition: Since unobserved heterogeneity is soaked up in the error term, that information is embodied in the Pooled OLS error.

Estimating $\boldsymbol{\Sigma}$, cont.

Strategy:

Step 2. The diagonal elements of $\Sigma = \sigma_\epsilon^2 + \sigma_c^2$ are estimated by

$$\hat{\sigma}_\epsilon^2 + \hat{\sigma}_c^2 = \frac{\hat{\mathbf{v}}'\hat{\mathbf{v}}}{N \times T - (K+1)} = \sigma_\epsilon^2 + \sigma_c^2 \qquad (23)$$

Step 3. To find σ_c^2, we can use off diagonal elements in the $E[vv'] = \hat{\mathbf{v}}\hat{\mathbf{v}}'$ within each cross section unit:

$$\hat{\sigma}_c^2 = \frac{1}{[N \times T(T-1)/2 - (K+1)]} \sum_{i=1}^{N} \sum_{t=1}^{T-1} \sum_{s=t+1}^{T} \hat{v}_{it}\hat{v}_{st} \qquad (24)$$

Intuition of Step 3.

There is information in $\hat{\mathbf{v}}\hat{\mathbf{v}}'$ that can help us pin down σ_c^2

Individual	*t*	*s*	*Row, Column*
1	1	2	1, 2
1	1	3	1, 3
1	2	3	2, 3
2	1	2	1, 2
2	1	3	1, 3
2	2	3	2, 3

(25)

$\hat{v}\hat{v}' =$

$$\begin{bmatrix} \hat{v}_{11}\hat{v}_{11} & \hat{v}_{11}\hat{v}_{12} & \hat{v}_{11}\hat{v}_{13} & \hat{v}_{11}\hat{v}_{21} & \hat{v}_{11}\hat{v}_{22} & \hat{v}_{11}\hat{v}_{23} \\ \hat{v}_{12}\hat{v}_{11} & \hat{v}_{12}\hat{v}_{12} & \hat{v}_{12}\hat{v}_{13} & \hat{v}_{12}\hat{v}_{21} & \hat{v}_{12}\hat{v}_{22} & \hat{v}_{12}\hat{v}_{23} \\ \hat{v}_{13}\hat{v}_{11} & \hat{v}_{13}\hat{v}_{12} & \hat{v}_{13}\hat{v}_{13} & \hat{v}_{13}\hat{v}_{21} & \hat{v}_{13}\hat{v}_{22} & \hat{v}_{13}\hat{v}_{23} \\ \hat{v}_{21}\hat{v}_{11} & \hat{v}_{21}\hat{v}_{12} & \hat{v}_{21}\hat{v}_{13} & \hat{v}_{21}\hat{v}_{21} & \hat{v}_{21}\hat{v}_{22} & \hat{v}_{21}\hat{v}_{23} \\ \hat{v}_{22}\hat{v}_{11} & \hat{v}_{22}\hat{v}_{12} & \hat{v}_{22}\hat{v}_{13} & \hat{v}_{22}\hat{v}_{21} & \hat{v}_{22}\hat{v}_{22} & \hat{v}_{22}\hat{v}_{23} \\ \hat{v}_{23}\hat{v}_{11} & \hat{v}_{23}\hat{v}_{12} & \hat{v}_{23}\hat{v}_{13} & \hat{v}_{23}\hat{v}_{21} & \hat{v}_{23}\hat{v}_{22} & \hat{v}_{23}\hat{v}_{23} \end{bmatrix} \tag{26}$$

Combining for Estimating β_{re}

Now, we can use our estimating equation for recovering the random effects estimates for β

$$\mathbf{b_{RE}} = \left(\mathbf{x}'\hat{\mathbf{\Omega}}^{-1}\mathbf{x}\right)^{-1}\mathbf{x}'\hat{\mathbf{\Omega}}^{-1}\mathbf{y} \tag{27}$$

Difference Between Random Effects and Pooled OLS

- Both RE and Pooled OLS are consistent (in large samples we expect estimated **b** to be equal to β).
- RE is efficient

Consider this experiment

$$y_{it} = \beta_0 + \beta_1 x_{it} + c_i + \epsilon_{it} = 1 + 1x_{it} + c_i + \epsilon_{it} \tag{28}$$

Draw values for $c_i \sim N(0, \sigma_c^2)$ and $\epsilon_{it} \sim N(0, \sigma_\epsilon^2)$ and **x** and calculate **y**, where T=3 and N=500.

Results

σ_c^2	σ_ϵ^2	Bias β_0 OLS	Bias β_0 RE	Bias β_1 OLS	Bias β_1 RE	Winner
5	1	-9.5%	-6.1%	.58%	.23%	RE
4	1	-1.22	-.16	.06	-.04	RE
3	1	-1.66	-1.6	-.12	-.13	RE
2	1	-2.22	-1.12	.24	.13	RE
1	1	.52	.46	-.06	-.05	RE
0	1	-.57	-.58	.004	.005	OLS

Note: Bias calculated as $\frac{b_k - \beta_k}{\beta_k} \times 100$ and averaged for 100 Monte Carlo Simulations.

Overview of Random Effects

- Assumes that Unobserved Heterogeneity lives in the model error term
- Since it is not estimated directly, we must assume that $E(\mathbf{x}'\mathbf{c}) = 0$. Pretty strong assumption.
- Allows the researcher to specify time-invariant independent variables (e.g. the individual's place of birth)- *The biggest advantage of Random Effects versus Fixed Effects*